Stokes Theorem

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Green's Theorem

Let D be an open domain in the plane $\mathbb{R}^2$ (think of $\mathbb{R}^2$ as xy plane in $\mathbb{R}^3$)

Suppose $\vec{F}$ is a vector field defined on D

ie $\vec{F} = P\hat{i} + Q\hat{j} + 0\hat{k}$ (will generalize to $R\hat{k}$)

and C is a simple closed curve in D s.t.

$\text{int}(C) \subseteq D$

Green Thm (Classical Form)

$$\int_C \vec{F}\cdot d\vec{r} = \iint_{\text{int}(C)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Green's Thm (Rewritten)

Think of int(C) as a surface in $\mathbb{R}^3$ with orientation upward (in z-dir.)

Let's consider a vector field $\vec{g}$ with $\vec{g}\cdot\vec{k} = \vec{k}$-component of $\vec{g}$.

$$= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Notice: $C = \partial(\text{int}(C))$ "boundary"

$$\int_{\partial(\text{in}(C))} \vec{F}\cdot d\vec{r} = \underbrace{\iint_{\text{int}(C)} \vec{g}\cdot d\vec{\sigma}}_{\text{surface integral}}$$

Note: For int(C) viewed as a surface in $\mathbb{R}^3$, its normal vector $\vec{n}$ is $\vec{k}$

$\Rightarrow \iint_{\text{int}(C)} \vec{g}\cdot d\vec{\sigma}$ depends only on $\vec{k}$ component of $\vec{g}$

Stokes Thm

Let D be a domain in $\mathbb{R}^3$.

Let Σ be a surface in D.

Suppose $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ is a vector field defined on D, differentiable

Then for an appropriate vector field $\vec{g}$ (determined by $\vec{F}$) with $\vec{k}\cdot\vec{g} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

We have:

$$\int_{\partial(\Sigma)} \vec{F}\cdot d\vec{r} = \iint_{\Sigma} \vec{g}\cdot d\vec{\sigma}$$

with consistent orientations via RHR

Remark

the RHS depends on Σ, while LHS depends only on $\partial\Sigma$

~~eg~~ Σ_1 = northern hemisphere of unit sphere

Σ_2 = southern hemisphere of unit sphere

then $\partial\Sigma_1 = \partial\Sigma_2$ = equator

therefore

$$\iint_{\Sigma_1} \vec{g}\cdot d\vec{\sigma} = \iint_{\Sigma_2} \vec{g}\cdot d\vec{\sigma}$$

caveat: may be $\pm$ depending on orientations

For ...

C_2

caveat: may be $\pm$ depending on orientations
For $(+)$, need both upward or both downward

This is similar to the statement that if C is a curve from P to Q, and $\vec{g} = \nabla F$, then

$$\int_C \vec{g} \cdot d\vec{r} = F(Q) - F(P)$$

$\rightarrow$ so the LHS depends only on the endpoints of C, (i.e. $\partial(C)$) and not a particular path btwn them

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What is $\vec{g}$ in terms of $\vec{F}$?

Recall

$$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$$

$$\vec{k} \cdot \vec{g} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Idea: pretend we have a "vector":

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

$$= \vec{g}$$

this is the $\vec{g}$ in terms of $\vec{F}$, making Stokes Thm true

Recall for $\vec{F} = P\hat{i} + Q\hat{j}$ in the plane, we defined

scalar fcn

$$\operatorname{curl} \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \longleftarrow \hat{k}\text{-component of curl } \vec{F} \text{ for } \vec{F} \text{ in } xy \text{ plane}$$

In fact the usual defn of curl is for 3-dim vector fields and produces another vector field, it is

$$\nabla \times \vec{F} = \operatorname{curl} \vec{F}$$

Statement of Stokes
For a vector field $\vec{F}$, surface Σ, all in some domain $\mathbb{R}^3$,

$$\int_{\partial\Sigma} \vec{F} \cdot d\vec{r} = \iint_{\Sigma} \operatorname{curl} \vec{F} \cdot d\vec{a}$$

Let Σ = disc of radius r around $P \in \mathbb{R}^3$, parallel to xy-plane
eg, if $P = (2,3,4)$, then this disc lies in the plane $z=4$
Let $P = (x_0, y_0, z_0)$

$$\int_{\partial\Sigma} \vec{F} \cdot d\vec{r} = \iint_{\Sigma} \operatorname{curl} \vec{F} \cdot d\vec{a}$$

$$= \iint_{\Sigma} (\operatorname{curl} \vec{F} \cdot \vec{k})\, dA$$

$$\approx (\operatorname{curl} \vec{F}(P) \cdot \vec{k})\operatorname{area}(\Sigma)$$

$\hookrightarrow$ As $r \to 0$, the $\approx$ gets better

$$\Rightarrow \operatorname{curl} \vec{F}(P) \cdot \vec{k} = \lim_{r \to 0} \frac{\int_{\partial\Sigma} \vec{F} \cdot d\vec{r}}{\operatorname{area}(\Sigma)}$$

Recall what about Σ gives the $\underbrace{\hat{k}\text{-component}}_{\text{bc } \Sigma \text{ is parallel to } xy\text{-plane}}$ of curl $\underbrace{\text{at } P}$?

bc Σ is a little disc around P
$\Rightarrow \partial\Sigma$ is a circle around P

Rewrite

Let $C_r^{xy}(P)$ = circle of radius r and center P lying in a plane parallel to xy-plane

then for any continuously diff'able vector field $\vec{F}$

$$\hat{k}\cdot \operatorname{curl} \vec{F}(P) = \lim_{r\to 0} \frac{\int_{C_r^{xy}(P)} \vec{F}\cdot d\vec{r}}{\pi r^2}$$

"curl as circulation"

Simil let $C_r^{yz}(P)$ = circle of radius r and center P lying in a plane parallel to yz-plane

$P = (x_0, y_0, z_0)$, then

$$C_r^{yz}(P) = \left\{ (x,y,z)\in\mathbb{R}^3 \;\middle|\; \begin{array}{l} x = x_0 \\ (y-y_0)^2 + (z-z_0)^2 = r^2 \end{array} \right\}$$

Then
$$\hat{i}\cdot \operatorname{curl} \vec{F}(P) = \lim_{r\to 0} \frac{\int_{C_r^{yz}(P)} \vec{F}\cdot d\vec{r}}{\pi r^2}$$

Similary

$$\hat{j}\cdot \operatorname{curl} \vec{F}(P) = \lim_{r\to 0} \frac{\int_{C_r^{xz}(P)} \vec{F}\cdot d\vec{r}}{\pi r^2}$$

Theorem $\operatorname{curl}(\nabla F) = 0$

Proof 1

write $\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$ then $\operatorname{curl}(\nabla F)$ using the fact that mixed partials commute

Heuristic if we think of $\vec{\nabla}$ as a vector:

∇F is like $\vec{\nabla}$ times scalar F

$\Rightarrow \nabla F$ is "parallel" to $\vec{\nabla}$

$\Rightarrow \vec{\nabla}\times(\vec{\nabla} F) = 0$

Proof 2

path integral of ∇F is path-ind (bc conservative)

$\Rightarrow \int_C \nabla F\cdot d\vec{r} = 0$ if C a closed curve

eg, for $C = C_r^{xy}(P), C_r^{yz}(P), C_r^{xz}(P)$

$\Rightarrow$ each component of $\operatorname{curl}(\nabla F)$ at any point P in the domain of F is

$$\lim \frac{0}{\;} = 0$$

domain of F is

$$\lim_{r \to 0} \frac{0}{\pi r^2} = 0$$

$\Rightarrow \text{curl}(\nabla F) = 0$